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A proof of the equivalence of two complex structures on the punctured tangent bundle of complex projective space

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Abstract

We give a proof of the equivalence of two complex structures on the punctured tangent bundle $\overset{\circ}{T}P^n(\mathbf{C})$ of complex projective space $P^n(\mathbf{C})$.

1 Introduction

K. Furutani, R. Tanaka, and S. Yoshizawa constructed a complex (Kähler) structure on the punctured (co-)tangent bundle $\overset{\circ}{T}P^n(\mathbf{C})$ of the complex projective space $P^n(\mathbf{C})$ by constructing a diffeomorphism of $\overset{\circ}{T}P^n(\mathbf{C})$ onto a "complex cone" $\mathcal{A}(\mathbf{C})$ in the space $M(n+1, \mathbf{C})$ of complex matrices ([FT], [FY]). Motivated by their works, we also constructed a complex (Kähler) structure on $\overset{\circ}{T}P^n(\mathbf{C})$ by a Riemannian geometric method ([IY], [Ii], [IM]). Since our complex structure has similar properties as that of Furutani et al., it seems that they coincide, but no proof has been given. The purpose of the present note is to prove that these structures coincide. (cf. [Na])

2 Tangent bundle of Riemannian manifold

Let M be a Riemannian manifold of dimension n with a Riemannian metric g. Let T_pM denote the tangent space to M at a point p of M, TM the tangent bundle of M, and π the bundle projection of TM onto M. Let T_uTM denote the tangent space to TM at $u \in TM$, and TTM the tangent bundle of TM. Let ∇ denote the Levi-Civita connection of M, and $\mathcal{K} : TTM \to TM$ the connection map corresponding to ∇. For each $u \in TM$,

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let $T_u^H TM$ (resp. $T_u^V TM$) denote the kernel of $\mathcal{K}|_{T_uTM}$ (resp. $d\pi|_{T_uTM}$), which is an n-dimensional subspace of T_uTM called the horizontal (resp. vertical) subspace of T_uTM. We have a direct-sum decomposition:

$$T_uTM = T_u^H TM \oplus T_u^V TM.$$

Elements of $T_u^H TM$ (resp. $T_u^V TM$) are called horizontal (resp. vertical) vectors at u. If $u, v \in T_pM$, v_u^H (resp. v_u^V) will denote the horizontal (resp. vertical) vector obtained by the horizontal (resp. vertical) lift of v to T_uTM. Note that v_u^H is characterized by

$$\mathcal{K}(v_u^H) = 0 \quad \text{and} \quad d\pi(v_u^H) = v,$$

and v_u^V is characterized by

$$\mathcal{K}(v_u^V) = v \quad \text{and} \quad d\pi(v_u^V) = 0.$$

Note that the standard almost complex structure J_0 on TM is a $(1,1)$-tensor field on TM characterized by

$$J_0(v_u^H) = v_u^V \quad \text{and} \quad J_0(v_u^V) = -v_u^H.$$

(cf. [Do], [GKM], [Sa1], [Sa2])

3 Complex projective space

Let $M(n{+}1, \mathbf{C})$ denote the space of $(n{+}1)\times(n{+}1)$ complex matrices with the bracket operation $[A, B] = AB - BA$, and the norm $\|A\| = \sqrt{\mathrm{tr}(A^*A)}$. Let $Herm(n{+}1)$ denote the subspace of $M(n{+}1, \mathbf{C})$, consisting of Hermitian matrices with the Euclidean inner product $(A, B) \mapsto \mathrm{tr}(AB)$. A vector tangent to $Herm(n{+}1)$ at a point $P \in Herm(n{+}1)$ is denoted by a pair (P, V) with $V \in Herm(n{+}1)$. The tangent space to $Herm(n{+}1)$ at P is denoted by

$$T_P Herm(n{+}1) = \{(P, V) \mid V \in Herm(n{+}1)\}.$$

The complex projective space $P^n(\mathbf{C})$ of complex dimension n is represented as a submanifold of $Herm(n{+}1)$ as follows:

$$P^n(\mathbf{C}) = \{P \in Herm(n{+}1) \mid P^2 = P,\ \mathrm{tr}P = 1\}.$$

The tangent space to $P^n(\mathbf{C})$ at a point $P \in P^n(\mathbf{C})$ is given by

$$T_P P^n(\mathbf{C}) = \{(P, Q) \mid Q \in Herm(n{+}1),\ PQ + QP = Q\}.$$

Let g denote the induced Riemannian metric on $P^n(\mathbf{C})$, i.e., $g((P, Q), (P, R)) = \mathrm{tr}(QR)$. The tangent bundle to $P^n(\mathbf{C})$ is denoted by $TP^n(\mathbf{C})$.

Lemma 3. 1 *The standard complex structure j on $P^n(\mathbf{C})$ is given by*

$$j : TP^n(\mathbf{C}) \to TP^n(\mathbf{C}), \quad j((P, Q)) = (P, \sqrt{-1}[P, Q]).$$

Let $\mathbf{C}^{n+1}$ be the complex $(n+1)$-space with the standard norm $\| \ \|$;

$$\mathbf{C}^{n+1} = \{ p = {}^t(p_1, p_2, \ldots, p_{n+1}) \mid p_i \in \mathbf{C} \}.$$

Lemma 3. 2 ([FT], [FY]) *For any point P of $P^n(\mathbf{C})$, and for any vector (P, Q) tangent to $P^n(\mathbf{C})$ at P, there exist $p, q \in \mathbf{C}^{n+1}$ which satisfy*

(1) $\|p\| = 1$, (2) $p^* q = 0$, (3) $P = pp^*$, (4) $Q = pq^* + qp^*$.

Note that $\|Q\|^2 = 2\|q\|^2$.

Lemma 3. 3 ([FT], [FY]) *Let (P, Q) and (P, R) be vectors tangent to $P^n(\mathbf{C})$ at P. Then we have*

(1) $\mathrm{tr} Q = \mathrm{tr}(PQ) = 0$,

(2) $PQP = 0$,

(3) $PQ^2 = Q^2 P = \dfrac{1}{2}\|Q\|^2 P$,

(4) $Q^2 = \dfrac{1}{2}\|Q\|^2 P + QPQ$,

(5) $Q^3 = \dfrac{1}{2}\|Q\|^2 Q$,

(6) $\|PQ\|^2 = \dfrac{1}{2}\|Q\|^2$,

(7) $\mathrm{tr}(PQ^2) = \dfrac{1}{2}\|Q\|^2$,

(8) $PQR = QRP$,

(9) $P(QR + RQ) = \mathrm{tr}(QR)P$,

(10) $Q(I - 2P) = -(I - 2P)Q$,

(11) $\mathrm{tr}([P, Q]Q) = 0$,

(12) $Q[P, Q] = -[P, Q]Q$,

(13) $\|Q\|^2 R + 2(I - 2P)[Q^2, R] = \mathrm{tr}(QR)Q - \mathrm{tr}([P, Q]R)[P, Q]$,

where I denotes the unit matrix.

$T_P P^n(\mathbf{C})$ is a linear subspace of the Euclidean space $T_P Herm(n+1)$.

Lemma 3. 4 *The orthogonal projection $\tau_P : T_P Herm(n+1) \to T_P P^n(\mathbf{C})$ is given by*

$$\tau_P(P, V) = (P, PV + VP - 2PVP) = (P, [P, [P, V]]).$$

Proof $(P, PV + VP - 2PVP)$ is tangent to $P^n(\mathbf{C})$ at P, since $(PV + VP - 2PVP)^* = PV + VP - 2PVP$, and $P(PV + VP - 2PVP) + (PV + VP - 2PVP)P = PV + VP - 2PVP$. $(P, V) - (P, PV + VP - 2PVP) = (P, V - (PV + VP - 2PVP))$ is orthogonal to each vector (P, Q) tangent to $P^n(\mathbf{C})$, since $\mathrm{tr}(Q(V - (PV + VP - 2PVP))) = 0$. ∎

Let $t \mapsto P(t)$ be a C^∞ curve on $P^n(\mathbf{C})$, and $\xi : t \mapsto \xi(t) = (P(t), Q(t))$ be a vector field along this curve such that $\xi(t)$ is tangent to $P^n(\mathbf{C})$ at $P(t)$. Then the covariant derivative $\nabla_{\frac{d}{dt}} \xi : t \mapsto \nabla_{\frac{d}{dt}} \xi(t)$ of ξ is defined by

$$\nabla_{\frac{d}{dt}} \xi(t) := \tau_{P(t)}(P(t), Q'(t)),$$

which is a vector field along this curve. ξ is said to be *parallel* if $\nabla_{\frac{d}{dt}} \xi(t) = (P(t), 0)$ for all t.

Lemma 3. 5 *Let $(P,Q),(P,R)$ be vectors tangent to $P^n(\mathbf{C})$ at P, and $t \mapsto P(t)$ be a C^∞ curve on $P^n(\mathbf{C})$ such that $P(0)=P$ and $\dot{P}(0) := (P(0),P'(0)) = (P,R)$. Let $\xi : t \mapsto \xi(t) = (P(t),Q(t))$ be the parallel vector field along this curve with initial condition $\xi(0) = (P(0),Q(0)) = (P,Q)$. Then we have*

$$Q'(0) = (I-2P)(QR+RQ).$$

Proof Since ξ is parallel, we have by Lemma 3. 4

$$(P(t),0) = \nabla_{\frac{d}{dt}}\xi(t) = (P(t), P(t)Q'(t)+Q'(t)P(t)-2P(t)Q'(t)P(t)).$$

Putting $t=0$, we have

$$PQ'(0)+Q'(0)P-2PQ'(0)P=0. \tag{1}$$

Multiplying (1) by P on the left, we have (since $P^2=P$)

$$PQ'(0)+PQ'(0)P-2PQ'(0)P=0. \tag{2}$$

Multiplying (1) by P on the right, we have

$$PQ'(0)P+Q'(0)P-2PQ'(0)P=0. \tag{3}$$

From (2) and (3) we have

$$PQ'(0)=Q'(0)P=PQ'(0)P. \tag{4}$$

On the other hand, since $\xi(t)$ is tangent to $P^n(\mathbf{C})$ at $P(t)$, we have $P(t)Q(t)+Q(t)P(t) = Q(t)$ for all t. Differentiating this with respect to t, and putting $t=0$, we have $P(0)Q'(0)+P'(0)Q(0)+Q(0)P'(0)+Q'(0)P(0)=Q'(0)$. Since $P(0)=P$, $Q(0)=Q$, and $P'(0)=R$, we have

$$Q'(0)=PQ'(0)+RQ+QR+Q'(0)P. \tag{5}$$

Multiplying (5) by P on the left, we have $PQ'(0)=PQ'(0)+PRQ+PQR+PQ'(0)P$. Hence

$$PQ'(0)P=-PQR-PRQ. \tag{6}$$

From (4), (5), and (6) we obtain

$$Q'(0)=QR+RQ-2PQR-2PRQ=(I-2P)(QR+RQ). \qquad \blacksquare$$

A vector tangent to $TP^n(\mathbf{C})$ at (P,Q) is denoted by $((P,Q),(R,V))$. The tangent space to $TP^n(\mathbf{C})$ at a point (P,Q) is given by

$$\begin{aligned} T_{(P,Q)}TP^n(\mathbf{C}) = \{((P,Q),(R,V)) \mid R,V \in Herm(n{+}1),\ & PR+RP=R, \\ & PV+VP+QR+RQ=V\}. \end{aligned}$$

The tangent bundle of $TP^n(\mathbf{C})$ is denoted by $TTP^n(\mathbf{C})$. Let $(P,Q),(P,R) \in T_PP^n(\mathbf{C})$. Let $t \mapsto P(t)$ be a C^∞ curve on $P^n(\mathbf{C})$ such that $P(0)=P$ and $\dot{P}(0) := (P(0),P'(0)) = (P,R)$, and $\xi : t \mapsto \xi(t) = (P(t),Q(t))$ be the parallel vector field along this curve with initial condition $\xi(0) = (P(0), Q(0)) = (P, Q)$. Then the horizontal lift $(P, R)^H_{(P,Q)}$ of (P, R) to $T_{(P,Q)}TP^n(\mathbf{C})$ is given by

$$(P, R)^H_{(P,Q)} = \dot{\xi}(0) := (\xi(0), \xi'(0)) = ((P(0), Q(0)), (P'(0), Q'(0))).$$

By Lemma 3. 5 , we have

$$(P, R)^H_{(P,Q)} = ((P, Q), (R, (I - 2P)(QR + RQ))).$$

The vertical lift $(P, R)^V_{(P,Q)}$ of (P, R) to $T_{(P,Q)}TP^n(\mathbf{C})$ is given by

$$(P, R)^V_{(P,Q)} = \dot{\eta}(0) := (\eta(0), \eta'(0)) = ((P, Q), (0, R)),$$

where η is a curve on $TP^n(\mathbf{C})$ given by $\eta(t) = (P, Q + tR)$. The horizontal subspace and the vertical subspace of $T_{(P,Q)}TP^n(\mathbf{C})$ are given, respectively, by

$$T^H_{(P,Q)}TP^n(\mathbf{C}) = \{((P, Q), (R, (I - 2P)(QR + RQ))) \mid (P, R) \in T_P P^n(\mathbf{C})\},$$

and

$$T^V_{(P,Q)}TP^n(\mathbf{C}) = \{((P, Q), (0, R)) \mid (P, R) \in T_P P^n(\mathbf{C})\}.$$

The differential $d\pi$ of the projection $\pi : TP^n(\mathbf{C}) \to P^n(\mathbf{C}),\ (P, Q) \mapsto P$ is given by

$$d\pi : TTP^n(\mathbf{C}) \to TP^n(\mathbf{C}), \quad d\pi((P, Q), (R, V)) = (P, R).$$

The connection map $\mathcal{K}$ is given by

$$\mathcal{K} : TTP^n(\mathbf{C}) \to TP^n(\mathbf{C}), \quad \mathcal{K}((P, Q), (R, V)) = (P, PV + VP - 2PVP).$$

Note that

(1) $d\pi((P, R)^H_{(P,Q)}) = (P, R)$, (2) $d\pi((P, R)^V_{(P,Q)}) = (P, 0)$,

(3) $\mathcal{K}((P, R)^H_{(P,Q)}) = (P, 0)$, (4) $\mathcal{K}((P, R)^V_{(P,Q)}) = (P, R)$.

Lemma 3. 6 *Let $((P, Q), (R, V)) \in T_{(P,Q)}TP^n(\mathbf{C})$. Then it is represented as a sum of horizontal and vertical vectors as follows:*

$$\begin{aligned}((P, Q), (R, V)) &= ((P, Q), (R, (I - 2P)(QR + RQ))) \\ &\quad + ((P, Q), (0, V - (I - 2P)(QR + RQ))) \\ &= (P, R)^H_{(P,Q)} + (P, V - (I - 2P)(QR + RQ))^V_{(P,Q)}.\end{aligned}$$

Proof It suffices to show that $(P, V - (I - 2P)(QR + RQ))$ is tangent to $P^n(\mathbf{C})$ at P, which follows easily from (8) of Lemma 3. 3 . ∎

4 Complex structure on $\overset{\circ}{T}P^n(\mathbf{C})$

Let $\overset{\circ}{T}P^n(\mathbf{C})$ denote the punctured tangent bundle of $P^n(\mathbf{C})$, i.e.

$$\overset{\circ}{T}P^n(\mathbf{C}) = \{(P, Q) \in TP^n(\mathbf{C}) \mid Q \neq 0\},$$

which is an open subset of $TP^n(\mathbf{C})$. We define an almost complex structure J on $\overset{\circ}{T}P^n(\mathbf{C})$ by

$$J((j(P,Q))^H_{(P,Q)}) = \sqrt{2}\|Q\|(j(P,Q))^V_{(P,Q)},$$
$$J((j(P,Q))^V_{(P,Q)}) = -\frac{1}{\sqrt{2}\|Q\|}(j(P,Q))^H_{(P,Q)},$$
$$J((P,R)^H_{(P,Q)}) = \frac{\|Q\|}{\sqrt{2}}(P,R)^V_{(P,Q)},$$
$$J((P,R)^V_{(P,Q)}) = -\frac{\sqrt{2}}{\|Q\|}(P,R)^H_{(P,Q)}$$

for (P,R) orthogonal to $j(P,Q)$, $((P,Q) \in \overset{\circ}{T}P^n(\mathbf{C}))$. It is known that J is integrable, i.e., J is a complex structure on $\overset{\circ}{T}P^n(\mathbf{C})$. (See [Ii], [IY], [IM].)

Definition 4. 1 ([FT], [FY]) *We define a complex submanifold $\mathcal{A}(\mathbf{C})$ of $M(n+1,\mathbf{C})$, and a mapping Φ by*

$$\mathcal{A}(\mathbf{C}) = \{A \in M(n+1,\mathbf{C}) \mid A^2 = 0,\ \mathrm{rank}A = 1\},$$
$$\Phi : \overset{\circ}{T}P^n(\mathbf{C}) \to \mathcal{A}(\mathbf{C}),\ \ \Phi(P,Q) = \|Q\|^2 P - Q^2 + \frac{\sqrt{-1}}{\sqrt{2}}\|Q\|Q.$$

Remark ([FT], [FY]) If we choose p, q as in Lemma 3. 2 , then

$$\Phi(P,Q) = (\|q\|p + \sqrt{-1}\,q)(\|q\|p - \sqrt{-1}\,q)^*.$$

Lemma 4. 2 ([FT], [FY]) *Φ is well-defined, and is a diffeomorphism.*

Lemma 4. 3 *Let $(P,Q) \in \overset{\circ}{T}P^n(\mathbf{C})$. Then we have*

(1) *For a tangent vector $((P,Q),(R,V))$ to $\overset{\circ}{T}P^n(\mathbf{C})$ at (P,Q),*

$$d\Phi((P,Q),(R,V)) = \Big(\Phi(P,Q),\ 2\mathrm{tr}(QV)P + \|Q\|^2 R - (QV+VQ) + \frac{\sqrt{-1}}{\sqrt{2}\|Q\|}\mathrm{tr}(QV)Q + \frac{\sqrt{-1}}{\sqrt{2}}\|Q\|V\Big).$$

(2) *For the horizontal lift to (P,Q) of a tangent vector (P,R),*

$$d\Phi((P,R)^H_{(P,Q)}) = \Big(\Phi(P,Q),\ \frac{1}{2}\mathrm{tr}(QR)Q - \frac{1}{2}\mathrm{tr}([P,Q]R)[P,Q] + \frac{1}{2}\|Q\|^2 R - \sqrt{-1}\sqrt{2}\|Q\|\mathrm{tr}(QR)P + \frac{\sqrt{-1}}{\sqrt{2}}\|Q\|(QR+RQ)\Big).$$

(3) *For the vertical lift to (P,Q) of a tangent vector (P,R),*

$$d\Phi((P,R)^V_{(P,Q)}) = \Big(\Phi(P,Q),\ 2\mathrm{tr}(QR)P - (QR+RQ) + \frac{\sqrt{-1}}{\sqrt{2}\|Q\|}\mathrm{tr}(QR)Q + \frac{\sqrt{-1}}{\sqrt{2}}\|Q\|R\Big).$$

(4) *For the horizontal lift to (P,Q) of $j(P,Q)$,*

$$d\Phi((j(P,Q))^H_{(P,Q)}) = \Big(\Phi(P,Q),\ \sqrt{-1}\|Q\|^2[P,Q]\Big).$$

(5) *For the vertical lift to* (P,Q) *of* $j(P,Q)$,

$$d\Phi((j(P,Q))^V_{(P,Q)}) = \left(\Phi(P,Q),\ -\frac{1}{\sqrt{2}}\|Q\|[P,Q]\right).$$

Lemma 4. 4 *Let* $(P,Q) \in \overset{\circ}{T}P^n(\mathbf{C})$, *and* (P,R) *be orthogonal to* $j(P,Q)$. *Then we have*

(1) $d\Phi(J((j(P,Q))^H_{(P,Q)})) = \sqrt{-1}\, d\Phi((j(P,Q))^H_{(P,Q)})$,

(2) $d\Phi(J((j(P,Q))^V_{(P,Q)})) = \sqrt{-1}\, d\Phi((j(P,Q))^V_{(P,Q)})$,

(3) $d\Phi(J((P,R)^H_{(P,Q)})) = \sqrt{-1}\, d\Phi((P,R)^H_{(P,Q)})$,

(4) $d\Phi(J((P,R)^V_{(P,Q)})) = \sqrt{-1}\, d\Phi((P,R)^V_{(P,Q)})$.

Proposition 4. 5 $$d\Phi \circ J = \sqrt{-1}\, d\Phi.$$

Theorem 4. 6 *If we identify* $\overset{\circ}{T}P^n(\mathbf{C})$ *with* $\mathcal{A}(\mathbf{C})$ *by* Φ, *then the complex srtucture* J *on* $\overset{\circ}{T}P^n(\mathbf{C})$ *coincides with the complex structure* $\sqrt{-1}$ *on* $\mathcal{A}(\mathbf{C})$.

Remark A similar result is obtained for the case of quaternion projective space. (cf. [Ka])

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